ESC194

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1 Complex Number Basic Properties

$$z = a + ib \quad i^2 = -1 \quad a, b, \text{ real numbers}$$

$$|z| \sqrt{a^2 + b^2} \rightarrow \text{modulus}$$

$$arg(z) \rightarrow \text{angle with real axis}$$

$$\bar{z} = a - ib \rightarrow \text{complex conjugate}$$

$$z \bar{+} w = \bar{z} + \bar{w}$$

Example:

$$(2 + 3i) + 4 - 5i = 6 - 2i$$

$$\overline{(2 + 3i)} + \overline{(4 - 5i)} = (2 - 3i) + 6 + 2i = \overline{6 - 2i}$$

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Example:

$$(2 + 3i)(1 - 2i) = 2 + 3i - 4i - 6i^2 = 8 - i$$

Complex Products Are:

- *Commutative*
- *Associative*
- *Distributive*

Therefore:

$$\overline{z_1 z_2} = \overline{z_1} + \overline{z_2}$$

Polar Form is useful:

$$z_1 = r_1(cos\theta + isin\theta) \qquad z_r = r_2(cos\theta + isin\theta)$$

where:

$$r_1 = |z_1| \qquad \theta = arg(z_1)$$

$$r_2 = |z_2| \qquad \theta = arg(z_2)$$

$$\therefore r_1 \cdot r_2 = r_1 \cdot r_2(cos\theta + isin\phi)(cos\theta + isin\phi)$$

$$\rightarrow |z_1 \cdot z_2| = |z_1|\,|z_2|$$

$$arg(z_1 \cdot z_2) = arg(z_1) + arg(z_2)$$

Product of many numbers:

$$|z_1 \cdot z_2 \cdot z_3....| = |z_1| \cdot |z_2|\,|z_3|\,....$$

$$arg(z_1 \cdot z_2 \cdot z_3...) = arg(z_1) + arg(z_2) + arg(z_3)...$$

Multiplying by I rotates 90 degrees ($\frac{\pi}{2}$)
Multiplying by -1 rotates 180 degrees (π)

2 De Moiure's Theorum

let

$$z = cos\theta + isin\theta$$

$$\rightarrow |z = 1|\,, arg(z) = \theta$$

$$|z^n| = |z|^n = 1$$

$$arg(z^n) = n \cdot arg(z) = n\theta$$

$$\therefore (cos\theta + isin\theta)^n = cos(n\theta) + isin(n\theta)$$

Example:

$$(cos\theta + isin\theta)^2 = cos(2\theta) + isin(2\theta)$$

$$LHS = (cos^2\theta - sin^2\theta) + i(2cos\theta sin\theta)$$

$$sin(2\theta) = 2sin(\theta)cos(\theta)$$

3 Complex Division

$$z^{-1} = \frac{1}{z} = \frac{1}{a+ib} = \frac{a-ib}{(a+ib)a-ib()} = \frac{a-ib}{a^2+b^2} = \frac{\overline{z}}{|z|^2}$$

$$\text{Since: } |\overline{z}| = |z| \qquad \text{and } arg(\overline{z}) = -arg(z)$$

$$\left|\frac{1}{z}\right| = \frac{|z|}{|z|^2} = \frac{1}{|z|}$$

$$arg(\frac{1}{z}) = -arg(z)$$

$$\frac{z}{w} = \frac{z\overline{w}}{|w|} = \frac{(a+ib)(c-id)}{c^2+d^2} = \frac{(ac+bd)+i(ad-cb)}{a^2+b^2}$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \text{ and } arg(\frac{z}{w}) = arg(z) - arg(w)$$

Example:

$$\frac{3+2i}{4-i} = \frac{(3+2i)(4+i)}{4^2+1^1} = \frac{(12-2)+i(3+8)}{17}$$

$$= \frac{10}{17} + i\frac{11}{17}$$

4 Complex Exponentials:

$$e^{ix} = ?$$

$$f(x) = e^{ix}$$

$$g(x) = cos(x) + isin(x)$$

$$f'(x) = ie^{ix}$$

$$= if(x)$$

$$g'(x) = -sin(x) + icos(x)$$

$$= ig(x)$$

$$f(0) = e^0 = 1$$

$$g(1) = cos(0) + isin(0) = 1$$

Leap of faith:

$$f(x) = g(x) \to e^{ix} = cos(x) + isin(x)$$

Euler's formula:

$$e^{i\theta} = cos(\theta) + i \cdot sin(\theta)$$

$$e^z = e^{a+ib} = e^a(cos(b) + isin(b))$$

Example:

$$e^{i\pi} = cos(\pi) + isin(\pi) = -1$$

5 17. Second-Order Differential Equations:

Second order Linear Equations:

$$p(x)y'' + q(x)y' + r(x)y = g(x)$$

We will be working with equations where p q and r are all constants.

$y''+ay'+by = g(x) \to$ 2nd order linear differential equation with constant coefficients.

$y''+ay'+by = 0 \to$ Homogeneous 2nd order linear DE with constant coefficients.

Theorum: If $y_1(x)$ and $y_2(x)$ are both solutions of a homogeneous 2nd order linear differential equation, and c_1, c_2 are any constants, then the linear combination:

$$y(x) = c_1y_1(x) + c_2y_2(x) \text{ is also a solution}$$

Proof: Not writing this down, who cares.

Theorem: If $y_1(x)$ and $y_2(x)$ are linearly independent, solutions to a homogeneous 2nd order linear differential equation, then:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \text{ is the general solution}$$

$$y_2(x) \neq A y_1(x) \rightarrow \text{ linearly independent}$$

Previously:

$$y' + ky = 0$$

Solution:

$$y = e^{-kx}$$

Now:

$$y'' + ay' + by = 0$$

Try:

$$y = e^{rx}$$

$$\therefore (e^{rx})'' + a(e^{rx})' + be^{rx} = 0$$

$$ar(r^2 + ar + b)e^{rx} = 0$$

That is a solution, only if the quadratic is zero, it's called the auxiliary, or characteristic equation:

$$\rightarrow r^2 + ar + b$$

or:

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Solutions depend heavily on what's going on in the determinant of the quadratic (what's under the square root).